

# LANDEN INEQUALITIES FOR ZERO-BALANCED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. For zero-balanced Gaussian hypergeometric functions  $F(a, b; a+b; x)$ ,  $a, b > 0$ , we determine maximal regions of  $ab$  plane where well-known Landen identities for the complete elliptic integral of the first kind turn on respective inequalities valid for each  $x \in (0, 1)$ . Thereby an exhausting answer is given to the open problem from [AVV].

KEYWORDS. Log-convexity; Hypergeometric functions; Inequalities.

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## 1. INTRODUCTION

Among special functions, the hypergeometric function has perhaps the widest range of applications. For instance, several well-known classes of mathematical physics are particular or limiting cases of it. For real numbers  $a, b$  and  $c$  with  $c \neq 0, -1, -2, \dots$ , the Gaussian hypergeometric function is defined by

$$(1.1) \quad F(a, b; c; x) := {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}$$

for  $x \in (-1, 1)$ , where

$$(a, n) := a(a+1)(a+2) \cdots (a+n-1)$$

for  $n = 1, 2, \dots$ , and  $(a, 0) = 1$  for  $a \neq 0$ . For many rational triples  $(a, b, c)$  the function (1.1) can be expressed in terms of elementary functions and long lists of such particular cases are given in [PBM].

It is clear that small changes of the parameters  $a, b, c$  will have small influence on the value of  $F(a, b; c; x)$ . In this paper we shall study to what extent some well-known properties of the complete elliptic integral of the first kind

$$(1.2) \quad \mathcal{K}(x) \equiv \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right) = \int_0^{\pi/2} (1 - x^2 \sin^2 t)^{-1/2} dt, \quad x \in (0, 1),$$

can be extended to  $F(a, b; a+b; x)$  for  $(a, b)$  close to  $(1/2, 1/2)$ . Recall that  $F(a, b; c; r)$  is called *zero-balanced* if  $c = a + b$ . In the zero-balanced case, there is a logarithmic singularity at  $r = 1$  and Gauss proved the asymptotic formula

$$(1.3) \quad F(a, b; a+b; r) \sim -\frac{1}{B(a, b)} \log(1-r)$$

as  $r$  tends to 1, where

$$(1.4) \quad B(z, w) \equiv \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad \operatorname{Re} z > 0, \operatorname{Re} w > 0$$

is the classical beta function. Note that  $\Gamma(1/2) = \sqrt{\pi}$  and  $B(\frac{1}{2}, \frac{1}{2}) = \pi$ , see ([AS, Ch. 6]).

Ramanujan found a much sharper asymptotic formula

$$(1.5) \quad B(a, b)F(a, b; a + b; r) + \log(1 - r) = R(a, b) + O((1 - r) \log(1 - r))$$

as  $r$  tends to 1 (see also [Ask1].) Here and in the sequel,

$$(1.6) \quad \begin{cases} R(a, b) \equiv -\Psi(a) - \Psi(b) - 2\gamma, & R(1/2, 1/2) = \log 16, \\ \Psi(z) \equiv \frac{d}{dz}(\log \Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}, & \operatorname{Re} z > 0, \end{cases}$$

and  $\gamma$  is the Euler-Mascheroni constant. Ramanujan's formula (1.5) is a particular case of another well-known formula given in ([AS, 15.3.10]).

We shall use in the sequel the following assertion which is a mixture of Biernacki-Krzyz and related results on the ratio of formal power series ([AVV],[BOR]).

**Lemma 1.7.** *Suppose that the power series  $f(x) = \sum_{n \geq 0} \hat{f}_n x^n$  and  $g(x) = \sum_{n \geq 0} \hat{g}_n x^n$  have the radius of convergence  $r > 0$  and  $\hat{g}_n > 0$  for all  $n \in \{0, 1, 2, \dots\}$ . Denote also*

$$h(x) = \frac{f(x)}{g(x)} = \sum_{n \geq 0} \hat{h}_n x^n.$$

1. *If the sequence  $\{\hat{f}_n/\hat{g}_n\}_{n \geq 0}$  is monotone increasing then  $h(x)$  is also monotone increasing on  $(0, r)$ .*

2. *If the sequence  $\{\hat{f}_n/\hat{g}_n\}_{n \geq 0}$  is monotone decreasing then  $h(x)$  is also monotone decreasing on  $(0, r)$ .*

3. *If the sequence  $\{\hat{f}_n/\hat{g}_n\}$  is monotone increasing (decreasing) for  $0 < n \leq n_0$  and monotone decreasing (increasing) for  $n > n_0$ , then there exists  $x_0 \in (0, r)$  such that  $h(x)$  is increasing (decreasing) on  $(0, x_0)$  and decreasing (increasing) on  $(x_0, r)$ .*

Some of the most important properties of the elliptic integral  $\mathcal{K}(r)$  are the Landen identities [WW, p.507]:

$$(1.8) \quad \mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), \quad \mathcal{K}\left(\frac{1-r}{1+r}\right) = \frac{1+r}{2}\mathcal{K}'(r),$$

where  $\mathcal{K}'(r) = \mathcal{K}(\sqrt{1-r^2})$ ,  $r \in (0, 1)$ . In [AVV, p.79], the following problem was raised:

**Open problem 1.9.** *Find an analog of Landen's transformation formulas in (1.8) for  $F(a, b; a + b; r)$ . In particular, if  $k(r) = F(a, b; a + b; r^2)$  and  $a, b \in (0, 1)$ , is it true that*

$$k(2\sqrt{r}/(1+r)) \leq Ck(r)$$

*for some constant  $C$  and all  $r \in (0, 1)$ ?*

Since  $2\sqrt{r}/(1+r) > r$  for  $r \in (0, 1)$ ,  $C$  must be greater than 1.

In [AVV, pp. 20-21] and [ABRVV, Theorem 1.4] Gauss' asymptotic formula (1.3) was refined by finding the lower and upper bounds for

$$W(r) = B(a, b)F(a, b; a + b; r) + (1/x) \log(1 - r),$$

when  $a, b \in (0, 1)$  or  $a, b \in (1, \infty)$ . Our second result gives a full solution to the Open Problem 1.9.

We wish to point out that in [QV, Thm 1.2(1)] it was claimed that for  $a, b \in (0, 1)$ ,  $c = a + b \leq 1$ , the function

$$(1.10) \quad s(r) = (1 + \sqrt{r})F(a, b; c; r) - F(a, b; c; 4\sqrt{r}/((1 + \sqrt{r})^2))$$

is increasing in  $r \in (0, 1)$ . As pointed out by A. Baricz [B] the proof contains a gap and the correct proof will be given here.

We also found another area in  $ab$  plane where the function  $s(r)$  is monotone decreasing in  $r \in (0, 1)$ .

## 2. MAIN RESULTS

Our first result shows that Landen inequalities hold not only in the neighborhood of the point  $a = b = 1/2$  but also in some unbounded parts of  $ab$  plane.

**Theorem 2.1.** *For all  $a, b > 0$  with  $ab \leq 1/4$  we have that the inequality*

$$F(a, b; a + b; 4r/(1 + r)^2) \leq (1 + r)F(a, b; a + b; r^2),$$

*holds for each  $r \in (0, 1)$ . Also, for  $a, b > 0$ ,  $1/a + 1/b \leq 4$ , the reversed inequality*

$$F(a, b; a + b; 4r/(1 + r)^2) \geq (1 + r)F(a, b; a + b; r^2),$$

*takes place for each  $r \in (0, 1)$ .*

*In the remaining region  $a, b > 0 \wedge ab > 1/4 \wedge 1/a + 1/b > 4$  neither of the above inequalities hold for each  $r \in (0, 1)$ .*

The disjoint regions in  $ab$  plane  $D_1 = \{(a, b) | a, b > 0, ab \leq 1/4\}$  and  $D_2 = \{(a, b) | a, b > 0, 1/a + 1/b \leq 4\}$ , where Landen inequalities hold, are shown on the Figure 1.

The only common point of the graphs in Figure 1 is  $(1/2, 1/2)$  where equality sign holds.

Two-sided bounds for the ratio of target functions are also possible.

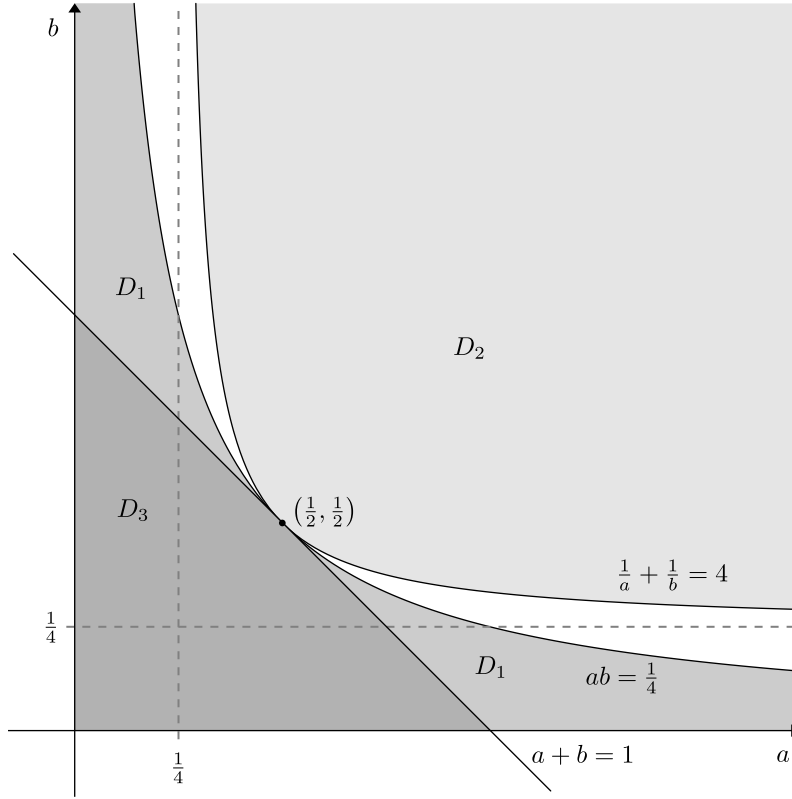
**Theorem 2.2.** *For each  $r \in (0, 1)$  and  $(a, b) \in D_1$ , we have*

$$1 < \frac{(1 + r)F(a, b; a + b; r^2)}{F(a, b; a + b; 4r/(1 + r)^2)} < \frac{B(a, b)}{\pi}.$$

*For  $(a, b) \in D_2$  the inequalities are reversed,*

$$\frac{B(a, b)}{\pi} < \frac{(1 + r)F(a, b; a + b; r^2)}{F(a, b; a + b; 4r/(1 + r)^2)} < 1.$$

Some numerical estimations of the constant  $C$  in Open Problem 1.9 follows.

FIGURE 1. The domains  $D_j, j = 1, 2, 3$  visualized.

**Corollary 2.3.** *Let  $k(\cdot)$  be defined as in the Open Problem 1.9. Then, for each  $r \in (0, 1)$  and  $(a, b) \in D_1$ , we have*

$$\frac{\pi}{B(a, b)} k(r) < k(2\sqrt{r}/(1+r)) < 2k(r)$$

*In the region  $D_2$  we have*

$$k(r) < k(2\sqrt{r}/(1+r)) < \frac{2\pi}{B(a, b)} k(r).$$

Two-sided bounds for the difference exist in a smaller region  $D_3 \subset D_1$  (see the picture), where  $D_3 = \{(a, b) | a, b > 0, a + b \leq 1\}$  and in  $D_2$ .

**Theorem 2.4.** *Let  $B = B(a, b)$  be the classical Beta function and  $R = R(a, b)$  be defined by 1.6.*

*For  $a, b > 0, a + b \leq 1$ , we have*

$$0 \leq (1 + \sqrt{r})F(a, b; a + b; r) - F(a, b; a + b; 4\sqrt{r}/((1 + \sqrt{r})^2)) \leq (R - \log 16)/B.$$

*If  $a, b > 0, 1/a + 1/b \leq 4$ , then*

$$0 \leq F(a, b; a + b; 4\sqrt{r}/((1 + \sqrt{r})^2)) - (1 + \sqrt{r})F(a, b; a + b; r) \leq (\log 16 - R)/B.$$

The second Landen identity has the following counterpart for hypergeometric functions. The resulting inequalities might be called Landen inequalities for zero-balanced hypergeometric functions.

**Theorem 2.5.** Let  $F(x) = F(a, b; a + b; x)$ .

For  $(a, b) \in D_1$  and each  $x \in (0, 1)$ , we have

$$\frac{1}{2} < \frac{F((\frac{1-x}{1+x})^2)}{(1+x)F(1-x^2)} < \frac{B(a, b)}{2\pi}.$$

If  $(a, b) \in D_3$ , then

$$(1+x)F(1-x^2) \leq 2F((\frac{1-x}{1+x})^2) \leq (1+x)[F(1-x^2) + (R - \log 16)/B].$$

For  $(a, b) \in D_2$ , we have

$$\frac{B(a, b)}{2\pi} < \frac{F((\frac{1-x}{1+x})^2)}{(1+x)F(1-x^2)} < \frac{1}{2},$$

and

$$0 \leq (1+x)F(1-x^2) - 2F((\frac{1-x}{1+x})^2) \leq (1+x)(\log 16 - R)/B.$$

### 3. PROOFS

Throughout this section we denote

$$F(x) = F(a, b; a + b; x), \quad G(x) = F(a, b; a + b + 1; x),$$

where  $a, b, (a, b) \neq (1/2, 1/2)$  are fixed positive parameters and

$$F_0(x) = F(1/2, 1/2; 1; x), \quad G_0(x) = F(1/2, 1/2; 2; x),$$

with the regions  $D_1, D_2, D_3$  defined as above.

The basic results, which makes possible all proofs in the sequel, are contained in the following

**Lemma 3.1.** 1. The function  $f(r) = F(r)/F_0(r)$  is monotone decreasing in  $r \in (0, 1)$  on  $D_1$  and monotone increasing on  $D_2$ .

2. The function  $g(r) = G(r)/G_0(r)$  is monotone decreasing on  $D_3$  and monotone increasing on  $D_2$ .

*Proof.* We shall use Lemma 1.7 in the proof.

Since  $\widehat{F}_n = (a)_n(b)_n/(a+b)_n(1)_n$ ,  $\widehat{F}_{0n} = ((1/2)_n/(1)_n)^2$ , applying the lemma one can see that the monotonicity of  $\{\widehat{F}_n/\widehat{F}_{0n}\}$  depends on the sign of

$$(3.2) \quad T_n = T(a, b; n) = n(ab - 1/4) + ab - (a+b)/4 = C_1n + C_2.$$

Since  $(a, b) \neq (1/2, 1/2)$  and

$$C_2 = \frac{\sqrt{ab}}{\sqrt{ab} + 1/2}C_1 - \frac{(\sqrt{a} - \sqrt{b})^2}{4},$$

it follows

1. If  $C_1 \leq 0$  i.e.  $(a, b) \in D_1$ , then  $C_2 < 0$ ; hence  $T_n < 0$  for  $n = 0, 1, 2, \dots$  and  $f(r)$  is monotone decreasing in  $r \in (0, 1)$ ;

2. If  $C_2 \geq 0$  i.e.  $(a, b) \in D_2$  then  $C_1 > 0$ , that is  $T_n > 0$ ,  $n = 0, 1, 2, \dots$  and  $f(r)$  is monotone increasing in  $r$ .

In the second case we have  $\widehat{G}_n = (a)_n(b)_n/(a+b+1)_n(1)_n$ ,  $\widehat{G}_{0n} = ((1/2)_n/(1)_n)^2/(n+1)$  and, proceeding analogously, we get

$$T_n = n(ab + a + b - 5/4) + 2ab - (a + b)/4 - 1/4 = C_3n + C_4.$$

3. If  $(a, b) \in D_3$ , that is  $a, b > 0, a + b \leq 1$ , let  $a + b = k > 0$ . Then  $ab \leq k^2/4$  and  $C_3 \leq k^2/4 + k - 5/4 = (k-1)(k+5)/4$ ;  $C_4 \leq k^2/2 - k/4 - 1/4 = (k-1)(2k+1)/4$ .

Since  $0 < k \leq 1$ , it follows that both  $C_3, C_4$  are non-positive. Therefore  $T_n < 0$ ,  $n = 0, 1, 2, \dots$  because both constants cannot be zero simultaneously. By Lemma 1.7, we conclude that the function  $g(r)$  is monotone decreasing in  $r \in (0, 1)$ .

4. If  $(a, b) \in D_2$ , i.e.,  $a, b > 0, 1/a + 1/b \leq 4$ , then  $4ab \geq a + b \geq 2\sqrt{ab}$ , hence  $ab \geq 1/4$ . Also  $a + b \geq 2\sqrt{ab} \geq 2 \cdot (1/2) = 1$ . Therefore  $C_3 \geq 0$  and  $C_4 = (ab - 1/4) + (4ab - a - b)/4 \geq 0$ . As above, we conclude that  $T_n > 0$ ,  $n = 0, 1, 2, \dots$  and  $g(r)$  is monotone increasing in this case. □

**3.3. Proof of Theorem 2.1.** By the above lemma, for each  $0 < x < y < 1$  we have  $f(x) > f(y)$  on  $D_1$  and  $f(x) < f(y)$  on  $D_2$ .

Putting  $x = x(r) = r^2$ ,  $y = y(r) = 4r/(1+r)^2$ , we get on  $D_1$ ,

$$\frac{F(r^2)}{F_0(r^2)} > \frac{F(y)}{F_0(y)},$$

that is, by Landen's identity,

$$F(y) < \frac{F_0(y)}{F_0(r^2)} F(r^2) = (1+r)F(r^2).$$

The second inequality is obtained analogously.

It is easily seen by (3.2) that in the remaining region the sequence  $\{\widehat{F}_n/\widehat{F}_{0n}\}$  decreases and then increases. By Lemma 1.7, part 3, this means that the function  $f(r)$ , for some  $r_0 \in (0, 1)$ , decreases in  $(0, r_0)$  and increases in  $(r_0, 1)$ . Therefore, putting  $0 < x(r) < y(r) < r_0$  and  $r_0 < x(r) < y(r) < 1$ , one concludes that neither of given inequalities hold for each  $r \in (0, 1)$ . □

**3.4. Proof of Theorem 2.2.** Since  $f(r)$  is monotone decreasing on  $D_1$ , applying Gauss formula, we obtain

$$1 = \lim_{r \rightarrow 0^+} \frac{F(r)}{F_0(r)} > \frac{F(r)}{F_0(r)} > \lim_{r \rightarrow 1^-} \frac{F(r)}{F_0(r)} = \frac{B(1/2, 1/2)}{B(a, b)} = \frac{\pi}{B(a, b)}.$$

Therefore,

$$\frac{F(y(r))}{F(x(r))} < \frac{B(a, b)}{\pi} \frac{F_0(y(r))}{F_0(x(r))} = (1+r) \frac{B(a, b)}{\pi},$$

by the Landen identity.

The inequality valid on  $D_2$  can be proved similarly. □

**3.5. Proof of Theorem 2.4.** Both assertions of this theorem are a consequence of the following

**Lemma 3.6.** *The function*

$$s(r) = (1 + \sqrt{r})F(a, b; a + b; r) - F(a, b; a + b; 4\sqrt{r}/((1 + \sqrt{r})^2))$$

*is monotone increasing in  $r \in (0, 1)$  on  $D_3$  and monotone decreasing on  $D_2$ .*

*Proof.* Let  $z = \frac{4\sqrt{r}}{(1+\sqrt{r})^2}$ . Then

$$1 - z = \frac{(1 - \sqrt{r})^2}{(1 + \sqrt{r})^2}; \quad \frac{dz}{dr} = \frac{2(1 - \sqrt{r})}{\sqrt{r}(1 + \sqrt{r})^3}.$$

Hence

$$\begin{aligned} s_1(r) &:= 2\sqrt{r}(1 - \sqrt{r})s'(r) = (1 - \sqrt{r})F(a, b; a + b; r) + 2\sqrt{r}(1 - r)F'(a, b; a + b; r) \\ &\quad - \frac{4}{1 + \sqrt{r}}(1 - z)F'(a, b; a + b; z) \\ &= (1 - \sqrt{r})F(a, b; a + b; r) + 2\frac{ab}{a + b}\sqrt{r}F(a, b; a + b + 1; r) - \frac{4ab}{(a + b)(1 + \sqrt{r})}F(a, b; a + b + 1; z) \\ &= (1 - \sqrt{r})F(r) + 2\frac{ab}{a + b}\sqrt{r}G(r) - \frac{4ab}{(a + b)(1 + \sqrt{r})}G(z). \end{aligned}$$

We used here the well-known formula

$$(3.7) \quad (1 - x)F'(a, b; a + b; x) = \frac{ab}{a + b}F(a, b; a + b + 1; x).$$

On the other hand, differentiating the first Landen identity we get

$$(3.8) \quad \frac{1}{1 + \sqrt{r}}G_0(z) = (1 - \sqrt{r})F_0(r) + \frac{1}{2}\sqrt{r}G_0(r).$$

Since  $g(r)$  is monotone decreasing on  $D_3$  and  $0 < r < z < 1$ , we get  $g(r) > g(z)$  i.e.,

$$G(z) < \frac{G_0(z)}{G_0(r)}G(r).$$

This, together with (3.8), yields

$$\begin{aligned} s_1(r) &> (1 - \sqrt{r})F(r) + 2\frac{ab}{a + b}\sqrt{r}G(r) - \frac{4ab}{(a + b)(1 + \sqrt{r})}\frac{G_0(z)}{G_0(r)}G(r) \\ &= (1 - \sqrt{r})F(r) + 2\frac{ab}{a + b}\sqrt{r}G(r) - \frac{4ab}{(a + b)}((1 - \sqrt{r})\frac{F_0(r)}{G_0(r)} + \frac{1}{2}\sqrt{r})G(r) \\ &= (1 - \sqrt{r})(F(r) - \frac{4ab}{(a + b)}\frac{F_0(r)}{G_0(r)}G(r)). \end{aligned}$$

By (3.7) again, we get

$$\frac{4ab}{(a + b)}\frac{G(r)}{G_0(r)} = \frac{F'(r)}{F'_0(r)}.$$

Hence,

$$2\sqrt{r}s'(r) > F(r) - \frac{F'(r)}{F'_0(r)}F_0(r) = \frac{F^2(r)}{F'_0(r)}\left(\frac{F_0(r)}{F(r)}\right)'.$$

The last expression is positive on  $D_3$  because  $D_3 \subset D_1$  and, by (3.1), the function  $f(r) = \frac{F(r)}{F_0(r)}$  is monotone decreasing on  $D_1$ .

Therefore we proved that the function  $s(r)$  is monotone increasing in  $r \in (0, 1)$  on  $D_3$ .

**Remark 3.9.** *Due to the remark in Introduction, this proof gives an affirmative answer to the 12 years old hypothesis risen in [QV].*

Since  $g(r)$  is increasing on  $D_2$ , we get

$$G(z) > \frac{G_0(z)}{G_0(r)} G(r).$$

Hence, proceeding as before, it follows that

$$2\sqrt{r}s'(r) < \frac{F^2(r)}{F_0'(r)} \left( \frac{F_0(r)}{F(r)} \right)' < 0,$$

since  $f(r) = \frac{F(r)}{F_0(r)}$  is monotone increasing on  $D_2$ .

Therefore  $s(r)$  is monotone decreasing in  $r \in (0, 1)$  on  $D_2$  and the proof of Lemma 3.6 is done.  $\square$

By Lemma 3.6 we obtain  $\lim_{r \rightarrow 0+} s(r) < s(r) < \lim_{r \rightarrow 1-} s(r)$  on  $D_3$  and  $\lim_{r \rightarrow 1-} s(r) < s(r) < \lim_{r \rightarrow 0+} s(r)$  on  $D_2$ .

Evidently,  $\lim_{r \rightarrow 0+} s(r) = 0$ .

Applying Ramanujan formula (1.5), we get

$$\begin{aligned} \lim_{r \rightarrow 1-} s(r) &= \lim_{r \rightarrow 1-} (R - 2\log(1 - r) + \log(1 - z) + o(1))/B \\ &= \lim_{r \rightarrow 1-} (R - 2\log(1 - \sqrt{r})(1 + \sqrt{r}) + 2\log \frac{1 - \sqrt{r}}{1 + \sqrt{r}} + o(1))/B = (R - \log 16)/B. \end{aligned}$$

The assertion of Theorem 2.4 follows.  $\square$

**3.10. Proof of Theorem 2.5.** Changing variable  $\frac{1-r}{1+r} = x \in (0, 1)$ , we obtain

$$r = \frac{1-x}{1+x}; \quad 1+r = \frac{2}{1+x}; \quad \frac{4r}{(1+r)^2} = 1-x^2.$$

Putting this in Theorems 2.2, 2.4, we obtain the assertions of Theorem 2.5.  $\square$

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